MATH 2050A: Mathematical Analysis I **Mid-Term Test**

Answer ALL Questions 31 Oct, 2017. 8:30-10:00

1. (i) Use ε-δ notation to show that $\lim_{x\to 1} \frac{1}{x^2+1} = \frac{1}{2}$.

   (ii) Use ε-δ notation to show that $\lim_{x\to\infty} \frac{\sin x^2}{1+x^2} = 0$.

   (iii) Does the limit $\lim_{x\to\infty} \frac{x^2 \sin x^2}{1+x^2}$ exist? Explain.

2. (i) State the Bolazno-Weierstrass Theorem and the Nested Intervals Theorem.

   (ii) Use the Bolazno-Weierstrass Theorem to show the Nested Interval Theorem.

3. Let $0 < a, b < 1$. Let $f : [0,1] \to [0,1]$ be a bijection. Suppose that $a|x-y| \leq |f(x) - f(y)| \leq b|x-y|$ for all $x, y \in [0,1]$.

   (i) Using ε-δ notation, show that f and the inverse f^{-1} both are continuous on $[0,1]$.

   (ii) Fix $x_1 \in [0,1]$. Put $x_{n+1} = f(x_n)$, for $n = 1, 2....$ Show that (x_n) is a Cauchy sequence.

   (iii) Show that there is a point $z \in [0,1]$ such that $f(z) = z$.

End

MATH2050A Midterm Solution

1. (i) $\left|\dfrac{1}{x^2+1}-\dfrac{1}{2}\right| = \left|\dfrac{1-x^2}{2(x^2+1)}\right| \leq \left|\dfrac{(1-x)(1+x)}{2}\right|$ for every $x \in \mathbb{R}$.

   If $0 < x < 2$, then $1 < 1+x < 3$ and $\left|\dfrac{1}{x^2+1}-\dfrac{1}{2}\right| \leq \dfrac{3}{2}|1-x|$

   Let $\epsilon > 0$. We put $\delta := \min(\frac{2}{3}\epsilon, 1)$. If $0 < |x-1| < \delta$, then

$$\left|\frac{1}{x^2+1}-\frac{1}{2}\right| < \frac{3}{2}\left(\frac{2\epsilon}{3}\right) = \epsilon$$

   (ii) $\left|\dfrac{\sin x^2}{1+x^2} - 0\right| \leq \dfrac{1}{1+x^2} \leq \dfrac{1}{x^2} \leq \dfrac{1}{x}$ whenever $x \geq 1$.

   Let $\epsilon > 0$. We put $M := \max(1, \frac{1}{\epsilon})$. If $x > M$, then

$$\left|\frac{\sin x^2}{1+x^2} - 0\right| \leq \frac{1}{x} < \frac{1}{M} \leq \epsilon$$

   (iii) No. Let $f(x) := \dfrac{x^2 \sin x^2}{1+x^2}$. Let $x_n := \sqrt{2n\pi}$ and $y_n := \sqrt{(2n+\frac{1}{2})\pi}$ for each $n \in \mathbb{N}$.

   Note $x_n, y_n \to \infty$ as $n \to \infty$, but $f(x_n) = 0$ for all n,

$$f(y_n) = \frac{(2n+\frac{1}{2})\pi}{1+(2n+\frac{1}{2})\pi} \to 1 \text{ as } n \to \infty$$

   Therefore, $\lim_{x\to\infty} f(x)$ does not exist.

   This follows from observing $f(x) = \left(\dfrac{x^2}{1+x^2}\right)\sin x^2$, $\lim_{x\to\infty} \dfrac{x^2}{1+x^2} \neq 0$ exists and $\lim_{x\to\infty} \sin x^2$ does not exist.

   One may also use $f(x) + \dfrac{\sin x^2}{1+x^2} = \sin x^2$ combining with 1(ii).

2. (i) The Bolzano-Weierstrass Theorem: A bounded sequence of real numbers has a convergent subsequence.

   Nested Intervals Theorem: If $\{I_n\}$ is a sequence of non-empty closed and bounded intervals such that $I_{n+1} \subset I_n$ for each $n \in \mathbb{N}$, then $\cap_{n=1}^{\infty} I_n \neq \emptyset$.

   Remark. No mark shall be given to the uniqueness part: Suppose further $(b_n - a_n) \to 0$ as $n \to \infty$, then $\cap_{n=1}^{\infty} I_n = \{\xi\}$ for some $\xi \in \mathbb{R}$, where $I_n = [a_n, b_n]$.

   (ii) Write $I_n = [a_n, b_n]$.

   Since each interval I_n is non-empty, pick $x_n \in I_n$. Note $a_1 \leq x_n \leq b_1$ and $\{x_n\}$ is a bounded sequence. By the Bolzano-Weierstrass Theorem, there exists a convergent subsequence $\{x_{n_k}\}$. Let $\xi \in \mathbb{R}$ be the limit of $\{x_{n_k}\}$. We claim that $\xi \in I_n$ for each $n \in \mathbb{N}$.

   Fix $N \in \mathbb{N}$. For $k \geq N$, $x_{n_k} \in I_{n_k} \subset I_k \subset I_N$ because $n_k \geq k \geq N$. Therefore, $a_N \leq x_{n_k} \leq b_N$ for every $k \geq N$. By taking limit $k \to \infty$, $a_N \leq \xi \leq b_N$. That is, $\xi \in I_N$ and this holds for every $N \in \mathbb{N}$.

3. (i) f is continuous on $[0,1]$: Fix any $x_0 \in [0,1]$. Let $\epsilon > 0$. We put $\delta := \frac{\epsilon}{b}$. If $x \in [0,1]$ with $|x - x_0| < \delta$, then $|f(x) - f(x_0)| \leq b\,|x - x_0| < b\left(\frac{\epsilon}{b}\right) = \epsilon$.

f^{-1} is continuous on $[0,1]$: Fix any $x_0 \in [0,1]$. Let $\epsilon > 0$. We put $\delta := a\epsilon$. If $x \in [0,1]$ with $|x - x_0| < \delta$, then

$$a\left|f^{-1}(x) - f^{-1}(x_0)\right| \leq \left|f(f^{-1}(x)) - f(f^{-1}(x_0))\right| = |x - x_0| < \delta = a\epsilon$$

hence $|f^{-1}(x) - f^{-1}(x_0)| < \epsilon$.

(ii) Observe that (x_n) is a contractive sequence satisfying

$$|x_{n+1} - x_n| = |f(x_n) - f(x_{n-1})| \leq b\,|x_n - x_{n-1}| \qquad \text{for every } n \geq 2$$

Hence, it is Cauchy: For any $n, p \in \mathbb{N}$, first by triangle inequality

$$\begin{aligned} |x_{n+p} - x_n| &\leq |x_{n+p} - x_{n+p-1}| + |x_{n+p-1} - x_{n+p-2}| + \dots + |x_{n+1} - x_n| \\ &\leq b^{p-1}\,|x_{n+1} - x_n| + b^{p-2}\,|x_{n+1} - x_n| + \dots + |x_{n+1} - x_n| \\ &= \left(b^{p-1} + b^{p-2} + \dots + 1\right)|x_{n+1} - x_n| \\ &\leq \frac{1}{1-b}\,|x_{n+1} - x_n| \\ &\leq \frac{b^{n-1}}{1-b}\,|x_2 - x_1| \end{aligned}$$

RHS is independent of p and tends to 0 as $n \to \infty$. Therefore, (x_n) is Cauchy.

(iii) Since (x_n) is Cauchy, $\lim_{n\to\infty} x_n$ exists. Let $z := \lim_{n\to\infty} x_n$ and check that it is the desired point. Since $0 \leq x_n \leq 1$ for each n, so is its limit z. By sequential criteria and continuity of f, $f(z) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} x_{n+1} = z$.